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On a System of Fuzzy Differential Inclusions

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Abstract. In this paper, two kinds of system of fuzzy differential inclusions are introduced and studied. An existence of the solutions for one system of fuzzy differential inclusions is proved by using continuous selection theorem. An existence of the solutions for another system of fuzzy differential inclusions is also proved by employing the fixed point theorem in the generalized metric space. The results presented in this paper improve and extend some known results concerned with the multivalued Cauchy problem and fuzzy differential inclusions.

1. Introduction

Since Zadeh [47] introduced the concept of fuzzy sets in 1965, the theory and application of fuzzy systems has been developed a lot in various aspects, especially in the theory of fuzzy control systems. For a fuzzy logic system, we usually apply fuzzy differential equations (FDEs) to analyze the performance of the system. With this method, the IF-THEN rules of the system could be embeded into FDEs. Therefore it is important to discuss the solution of fuzzy differential equations[5, 6]. Fuzzy-valued map were firstly developed by Puri and Ralescu [41]. They generalized and extended the concept of Hukuhara differentialbility (H-derivative) for a set-valued maps to fuzzy maps. After that, Kaleva [27] and Seikkala [43] started to develop a theory for fuzzy differential equtions(FDE) based on this idea. However, as the quite different properties between the solutions of fuzzy differential equations and crisp cases, this approach has some disadvantages that the diameter of the solution $x(t)$ is unbounded as the time t increases [18]. Hüllermeier [23] overcame this problem by replacing the FDE by a family of differential inclusions

$$\begin{cases} x'\beta \in F\beta(t, x\beta(t)), \quad 0 \le \beta \le 1 \\ x\beta(t_0) \in [X_0]\beta \end{cases}$$

This approach was exploited by Diamond in [15–18]. Above all, we can see that the study of FDEs has received much attention, and with the references there are many works being done for the solution of the FDEs in different cases [7, 9, 11, 12, 28–33, 48].

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In 2011, Choudary [13] pointed out that the proof of the celebrated theorem of Nieto [36] is not true, which makes it still an open question that under what righthand side conditions a FDE has a solution. As we know, the solution of a differential equation with discontinuous righthand side could be obtained in the sense of differential inclusions [20], which makes it interesting to discuss the FDEs through Fuzzy differential inclusions (FDIs).

Moreover, it is necessary to treat the qualitative properties for a complex system. For FDEs, most of the discussions on qualitative problems are based on the idea of Hüllermeier [21]. Therefore, it is important to discuss the solutions of fuzzy differential inclusions for the qualitative theory of FDEs.

Fuzzy differential inclusions were first presented by Baidosov [8]. Aubin [3] and Dordan [19] discussed the viability theory of the FDIs with toll sets. Lakshmikantham [31] presented a theorem to show that under what conditions the attainable sets of FDIs are the level sets of a fuzzy map. In 2000, Zhu and Rao [49] introduced a class of differential inclusions for fuzzy maps as follows

$$\begin{cases} F_{(t,x(t))}(x'(t)) > (\text{or } \geq)\alpha(x(t)), \\ \text{i.e. } x'(t) \in (F_{(t,x(t))})_{\alpha(x(t))}(\text{or } [F_{(t,x(t))}]_{\alpha(x(t))}), \\ x(t_0) = x_0 \end{cases}$$

for some function $\alpha : E \to [0,1]$, where E is a Banach space.

The FDIs in [15, 23, 31] are actually a class of parameterized differential inclusions. The FDIs in Zhu's work [49] make it possible to discuss the system in time-varying cases, that is, the dependence of the velocity of state on the system is time-varying. We should notice that the idea of Zhu is equivalent to Dordan's method in the sense of toll sets, thus the tools and results of differential inclusions and viability theories could be employed in FDIs. Moreover, for a fuzzy control system

$$\begin{cases} x'(t) = f(x(t), u(t)), \\ y = g(x(t), u(t)), \quad u(t) \in U(t) \end{cases}$$

the state x, the output y, and even the description of the system f are all probably fuzzy because of the uncertainty of the system, but the system control u has to be crisp for the sake of realizability. One can easily see that it is more feasible to get crisp control u of the system by selection theorems in sense of FDIs.

In this paper, we introduce and study two types of systems of fuzzy differential inclusions. We show an existence of the solutions for one of the systems of FDIs by using continuous selection theorem. We also show an existence of the solutions for the other system of FDIs by employing the fixed point theorem in the generalized metric spaces. The results presented in this paper improve and extend some known results concerned with the fuzzy initial value problem studied by Hüllermeier [23], the fuzzy differential inclusions discussed by Zhu and Rao [49] and the multivalued Cauchy problem considered by Petrusel [40].

2. Preliminaries

In this section, we will present some definitions and results which might be utilized in this paper.

Definition 2.1. *A fuzzy subset A of $\mathbf{R}^n$ is defined with its membership function, $\mu_A : \mathbf{R}^n \to [0,1]$. Then the fuzzy set A could be rewritten as $\{(x, \mu_A) | \quad x \in \mathbf{R}^n\}$. The α-open level set of A is denoted by*

$$(A)\alpha = \{x | \mu_A(x) > \alpha\}$$

and the α-closed level set of A is denoted by

$$[A]\alpha = \{x | \mu_A(x) \geq \alpha\}$$

for $\alpha \in [0,1)$ and $\alpha \in [0,1]$, respectively.

Sometimes, we do not distinguish a fuzzy set A and its membership function μ_A, that is, the membership of $x \in \mathbf{R}^n$ could be simply written as $A(x)$.

Denote $\mathbf{E}^n = \{u : \mathbf{R}^n \to [0,1] | \ u$ satisfies 1)- 4) below$\}$, where

1) u is *normal*, that is, there exists an $x \in \mathbf{R}^n$ such that $u(x_0) = 1$;

2) u is *fuzzy convex*, that is, for any $x, y \in \mathbf{R}^n$ and $\lambda \in [0,1]$, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$, which is equivalent with that for any $\alpha \in [0,1]$, $[u]\alpha = \{x| \ u(x) \geq \alpha\}$ is a convex set[47];

3) u is *upper semicontinuous*, that is for any α, $[u]\alpha$ is a closed set;

4) $[u]_0 = \overline{\{x \in \mathbf{R}^n | \ u(x) \geq 0\}}$ is compact.

It is obviously that, for each $u \in \mathbf{E}^n$, the α-closed level sets $[u]\alpha$ are elements in $P_k(\mathbf{R}^n)$, the family of the compact and convex subsets of $\mathbf{R}^n$. Then each u in $\mathbf{E}^n$ corresponds a family of subsets $\{[u]\alpha\}_{\alpha=0}^{1}$, in $P_k(\mathbf{R}^n)$, and we have a representation theorem below for this corresponding relation. The discussion in this paper about fuzzy sets are all based on $\mathbf{E}^n$.

Lemma 2.2. [37] *If $u \in \mathbf{E}^n$, then*

(i) $[u]\alpha \in P_k(\mathbf{R}^n)$ *for all* $0 \leq \alpha \leq 1$;

(ii) $[u]_{\alpha_2} \subset [u]_{\alpha_1}$ *for all* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

(iii) If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then $[u]\alpha = \bigcap_{k\geq 1}[u]_{\alpha_k}$;

Conversely, if $\{A\alpha| \ 0 \leq \alpha \leq 1\}$ is a family of subsets of $\mathbf{R}^n$ satisfying i)- iii) mentioned above, then there exists a $u \in \mathbf{E}^n$ such that $[u]\alpha = A\alpha$ and $[u]_0 = \overline{\bigcup_{0\leq\alpha\leq 1} A\alpha} \subset A_0$.

Let (X, d) be a metric space, $Y \subset X, x \in X$ and $\varepsilon > 0$, and we have the notations below.

$$\begin{aligned} D(Y,x) &= D(x,Y) = \inf\{d(x,y)| \ y \in Y\}, \\ B_X(x,\varepsilon) &= \{y \in Y| \ d(x,y) < \varepsilon\}, \\ V(Y,\varepsilon) &= \{x \in X| \ D(Y,x) \leq \varepsilon\}, \\ P_{cl}(X) &= \{Y \subset X | Y \text{ is nonempty closed in } X\}. \end{aligned}$$

If X is a linear normed space, $\|\cdot\|$ is the norm, let

$$P_k(X) = \{Y \subset X | Y \text{ is nonempty compact and convex in } X\}.$$

For any $A, B \in P_k(X)$, we define the metric on $P_k(X)$ as follows:

$$H(A,B) = \inf\{\varepsilon > 0 | A \subset V(B,\varepsilon), B \subset V(A,\varepsilon)\}.$$

It is easy to see that the pair $(P_k(X), H)$ is a complete metric space and H is the Hausdorff distance induced by d. Based on the metric defined above, the metric on $\mathbf{E}^n$ could be defined with the same notation. For any $u, v \in \mathbf{E}^n$, let

$$H(u,v) = \sup_{\alpha\in[0,1]} H([u]\alpha, [v]\alpha).$$

Then we know that $(\mathbf{E}^n, H)$ is a complete metric space [42]. Therefore, the continuity of fuzzy functions and the convergence of fuzzy sets can be discussed based on the metric H here.

Definition 2.3. *Let (X, d) be a a metric space and $T : X \to 2^X$ be a multivalued map. Then $x^* \in X$ is called a fixed point for T if $x^* \in T(x^*)$. The set of all fixed points will be denoted by $FixT$.*

Definition 2.4. *[25] Let $\mathcal{A}, \mathcal{B}$ be two metric spaces. A multivalued map $T : \mathcal{A} \to 2^{\mathcal{B}}$ is called upper semicontinuous at $x_0 \in \mathcal{A}$ if and only if for any neighborhood U of $T(x_0)$, there exists a neighborhood V of x_0 such that for each $x \in V$ we have $T(x) \subset U$. T is said to be upper semicontinuous (u.s.c.) on $\mathcal{A}$ if it is u.s.c. at any point $x_0 \in \mathcal{A}$.*

Remark 2.5. *A fuzzy valued map $F : X \to \mathbf{E}^n$ can generate a real valued function $\widetilde{F} : X \times \mathbf{R}^n \to [0,1]$, where for any $x \in X$, $y \in \mathbf{R}^n$, $\widetilde{F}(x,y) = F(x)(y)$. In the following discussion we do not distinguish F and $\widetilde{F}$ for the convenience, and denote $F(x)$ by $F_{(x)}$.*

In this paper we denote an open subset in $\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^m$ by Ω with $(t_0, x_0, y_0) \in \Omega$. A fuzzy map $F : \Omega \to \mathbf{E}^n$ is called *convex* if for every $(t, x, y) \in \Omega$, $a, b \in \mathbf{R}^n$ and $\lambda \in [0,1]$, we have

$$F_{(t,x,y)}(\lambda a + (1-\lambda)b) \geq \min\{F_{(t,x,y)}(a), F_{(t,x,y)}(b)\}.$$

Let $\alpha : \mathbf{R}^n \times \mathbf{R}^m \to [0,1)$ and $\beta : \mathbf{R}^n \times \mathbf{R}^m \to [0,1)$ (or $[0, 1]$) be two functions and I an closed interval in $\mathbf{R}$. We consider the following fuzzy differential inclusions: for fuzzy map $F : \Omega \to \mathbf{E}^n$ and $G : \Omega \to \mathscr{F}_b(\mathbf{R}^m)$, finding $x \in C(I, \mathbf{R}^n)$ and $y \in C(I, \mathbf{R}^m)$ such that

$$\begin{cases} F_{(t,x(t),y(t))}(x'(t)) > \alpha(x(t), y(t)), \\ G_{(t,x(t),y(t))}(y'(t)) > \beta(x(t), y(t)), \\ x(t_0) = x_0, y(t_0) = y_0 \end{cases} \tag{1}$$

and

$$\begin{cases} F_{(t,x(t),y(t))}(x'(t)) \geq \alpha(x(t), y(t)), \\ G_{(t,x(t),y(t))}(y'(t)) \geq \beta(x(t), y(t)), \\ x(t_0) = x_0, y(t_0) = y_0, \end{cases} \tag{2}$$

It is easy to see that problems (1) and (2) are equivalent to the following problems

$$\begin{cases} x'(t) \in (F_{(t,x(t),y(t))})_{\alpha(x(t),y(t))}, \\ y'(t) \in (G_{(t,x(t),y(t))})_{\beta(x(t),y(t))}, \\ x(t_0) = x_0, y(t_0) = y_0 \end{cases}$$

and

$$\begin{cases} x'(t) \in [F_{(t,x(t),y(t))}]_{\alpha(x(t),y(t))}, \\ y'(t) \in [G_{(t,x(t),y(t))}]_{\beta(x(t),y(t))}, \\ x(t_0) = x_0, y(t_0) = y_0, \end{cases}$$

respectively.

3. Solutions of Fuzzy Differential Inclusion (1)

Lemma 3.1. *(Yannelis-Prabhaker)* [46] *Let D be a paracompact Hausdorff topological space, Y be a topological vector space and $F : D \to 2^Y$ be a multifunction with nonempty convex values. If F has open lower sections, that is, for any $y \in Y$, $F^{-1}(y) = \{x \in D |\ y \in F(x)\}$ is open in D, then there exists a continuous function $f : D \to Y$ such that $f(x) \in F(x)$ for any $x \in D$.*

Definition 3.2. [49] *Let $F : \Omega \to \mathbf{E}^n$ be a fuzzy map. The map F is called lower open if $F_{(t,x,y)}(z)$ is lower semicontinuous in $(t, x, y) \in \Omega$.*

Theorem 3.3. *Let Ω be an open subset in $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m$ with $(t_0, x_0, y_0) \in \Omega$. Suppose that $F : \Omega \to \mathbf{E}^n$ and $G : \Omega \to \mathscr{F}_b(\mathbf{R}^m)$ are convex and lower open fuzzy maps and that $\alpha : \mathbf{R}^n \times \mathbf{R}^m \to [0,1)$ and $\beta : \mathbf{R}^n \times \mathbf{R}^m \to [0,1)$ are upper semicontinuous functions. Moreover, if F and G satisfy the condition below for some $L > 0$,*

$$\sup_{a_1\in[F(t,x,y)]_0, b_1\in[G(t,x,y)]_0} \ \sup_{a_2\in[F(t,u,v)]_0, b_2\in[G(t,u,v)]_0} \|(a_1, b_1) - (a_2, b_2)\|_{\mathbf{R}^{n+m}} \leq L\|(x, y) - (u, v)\|_{\mathbf{R}^{n+m}}$$

on some neighborhood $I = (\omega_-, \omega_+)$ for any x, y, u and v, then there exists two continuous differentiable functions $x : I \to \mathbf{R}^n$ and $y : I \to \mathbf{R}^m$, which are the solution of problem (1).

Proof. We define a multifunction $\widetilde{F} : \Omega \to 2^{\mathbf{R}^n}$ by

$$\widetilde{F}(t,x,y) = (F_{(t,x,y)})_{\alpha(x,y)}, \quad (t,x,y) \in \Omega.$$

As the values of fuzzy map F belongs to $\mathbf{E}^n$, $\widetilde{F}(t,x,y)$ is always nonempty. For any $a, b \in \widetilde{F}(t,x,y)$ and $\lambda \in [0,1]$, by the convexity of F, we have

$$F_{(t,x,y)}(\lambda a + (1-\lambda)b) \geq \min\{F_{(t,x,y)}(a), F_{(t,x,y)}(b)\} > \alpha(x,y)$$

and so

$$\lambda a + (1-\lambda)b \in \widetilde{F}(t,x,y),$$

which means that $\widetilde{F}(t,x,y)$ is convex for each $(t,x,y) \in \Omega$. To utilize Lemma 3.1, we need only to prove that $\widetilde{F}$ has open lower sections. For any $z \in \mathbf{R}^n$,

$$\widetilde{F}^{-1}(z) = \{(t,x,y) \in \Omega |\, z \in \widetilde{F}(t,x,y)\} = \{(t,x,y) \in \Omega |\, F_{(t,x,y)}(z) > \alpha(x,y)\}.$$

Denote the complement set of $\widetilde{F}^{-1}(z)$ by

$$N \triangleq \{(t,x,y) \in \Omega |\, F_{(t,x,y)}(z) \leq \alpha(x,y)\}.$$

Then it is sufficient to show that N is a closed set. In fact, for any given a sequence $\{(t_n, x_n, y_n)\} \subset N$ with $(t_n, x_n, y_n) \to (t,x,y)$, we have

$$F_{(t_n,x_n,y_n)}(z) \leq \alpha(x_n, y_n).$$

By the lower open property of F and the upper semicontinuity of α, we obtain

$$F_{(t,x,y)}(z) \leq \liminf_{n\to\infty}(z) \leq \limsup_{n\to\infty} \alpha(x_n, y_n) \leq \alpha(x,y),$$

which means $(t,x,y) \in N$ and N is a closed set. From Lemma 3.1, there exists a continuous function $f : \Omega \to \mathbf{R}^n$ such that $f(t,x,y) \in \widetilde{F}(t,x,y)$ for each $(t,x,y) \in \Omega$.

Similarly, we can also define the other multifunction $\widetilde{G}(t,x,y) = (G_{(t,x,y)})_{\beta(x,y)}$, and get the continuous selection $g(t,x,y) \in \widetilde{G}(t,x,y)$ for each $(t,x,y) \in \Omega$ with the same method. Thus, the existence of the solution of Cauchy problem

$$\begin{cases} x'(t) = f(t, x(t), y(t)), \\ y'(t) = g(t, x(t), y(t)), \\ x(t_0) = x_0, y(t_0) = y_0 \end{cases}$$

under the Lipschitz condition

$$\begin{aligned} & \|(f(t,x,y), g(t,x,y)) - (f(t,u,v), g(t,u,v))\|_{\mathbf{R}^{n+m}} \\ \leq & \sup_{a_1 \in [F(t,x,y)]\alpha, b_1 \in [G(t,x,y)]\beta} \ \sup_{a_2 \in [F(t,u,v)]\alpha, b_2 \in [G(t,u,v)]\beta} \|(a_1, b_1) - (a_2, b_2)\|_{\mathbf{R}^{n+m}} \\ \leq & \sup_{a_1 \in [F(t,x,y)]_0, b_1 \in [G(t,x,y)]_0} \ \sup_{a_2 \in [F(t,u,v)]_0, b_2 \in [G(t,u,v)]_0} \|(a_1, b_1) - (a_2, b_2)\|_{\mathbf{R}^{n+m}} \\ \leq & L\|(x,y) - (u,v)\|_{\mathbf{R}^{n+m}} \end{aligned}$$

on some interval $I = [\omega_-, \omega_+]$ is a classical result. Therefore, there exists two continuous differentiable functions $x : I \to \mathbf{R}^n$ and $y : I \to \mathbf{R}^m$, which are the solution of problem (1). This completes the proof. □

Remark 3.4. *Theorem 3.3 extends Theorem 1 of Zhu and Rao* [49] *from a differential inclusion to the system of fuzzy differential inclusions.*

4. Solutions of Fuzzy Differential Inclusion (2)

In this section, to get the solution of problem (2), we need some more definitions and lemmas. We should notice that the metric defined in the proof of the following Theorem 4.8 is not the normal case, which makes us have to employ the concept and properties of the generalized metric space. The concept of the generalized metric space was introduced by Luxemburg [38] and Jung [24] as follows:

Definition 4.1. *The pair (X, d) will be called a generalized metric space if X is an arbitrary nonempty set and d is a function $d : X \times X \to [0, +\infty]$ which fulfills all the standard conditions for a metric.*

A generalized metric space is complete if all the Cauchy sequences in this space converge.

Definition 4.2. *Let (X, d) be a generalized metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. A sequence $(x_n)_{n\in\mathbf{N}} \subset X$ is called the sequence of successive approximations of T if and only if $x_0 \in X$ and $(x_n) \in T(x_{n-1})$ for all $n \in \mathbf{N} \cup \{0\}$.*

Lemma 4.3. [2] *Let $\Omega \subset R \times R^n \times R^m$ be an open set, $(t_0, x_0, y_0) \in \Omega$ and $F : \Omega \to P_k(R^n \times R^m)$ an upper semicontinuous multivalued operator. Then there exist $I = [t_0 - a, t_0 + a] \subset R$ (where $a > 0$) and $M > 0$ such that*

(i) $I \times B_{R^n\times R^m}((x_0, y_0), aM) \subset \Omega$;

(ii) $\|F(t, x, y)\| \leq M$ *on* $I \times B_{R^n\times R^m}((x_0, y_0), aM)$.

Lemma 4.4. [10] *Let X be a Banach space and $V, U : I \to P(X)$ be two multivalued measurable operators with compact values. If $v(t) \in V(t)$ is a measurable selection, then there is a measurable selection $u(s) \in U(t)$ such that*

$$\|u(t) - v(t)\| \leq H(U(t), V(t)), \quad \forall t \in I.$$

Lemma 4.5. [39] *Let (Z, d) be a complete generalized metric space and $T : Z \to P_{cl}(Z)$ be a multivalued α-contraction. Suppose that there is a sequence $(z_n)_{n\in N} \subset Z$ of successive approximations of T such that there exists an index $N_0 \in \mathbf{N}$ with the following property: $d(z_{N_0}, z_{N_0+l}) < \infty$, for all $l \in \mathbf{N} \cup \{0\}$. Then $Fix T \neq \emptyset$.*

Definition 4.6. *Let $(S, \mathcal{A}, \mu)$ be a complete σ−finite measure space and $(X, \|\cdot\|)$ be a separable Banach space. A multivalued operator $T : S \to P_{cl}(X)$ is said to be integrably bounded if and only if there is a function $r \in L^1(S)$ such that for all $v \in T(s)$ we have $\|v\| \leq r(s)$ a.e.*

For $1 \leq p \leq \infty$, we define the set

$$S^p_T = \{f \in L^p(\Omega, X) | f(s) \in T(s), a.e.\},$$

i.e., S^p_T contains all selections of T that belong to Lebesgue-Bochner space $L^p(\Omega, X)$. It is obvious that S^1_T is a closed subset of $L^1(\Omega, X)$ and it is nonempty if and only if T is integrably bounded [2, 4, 22].

A fuzzy map $F : I \to \mathbf{E}^n$ is called *fuzzy integrably bounded* if $F_0(t) = [F(t)]_0$ is integrably bounded.

Definition 4.7. *A fuzzy map $F : \Omega \to \mathbf{E}^n$ is called α-level uniformly continuous, if given a positive number ε, there exists a δ such that for any $(t, x, y) \in \Omega$, as long as $\alpha, \beta \in [0, 1]$ meet the condition $|\beta - \alpha| < \delta$, we have*

$$H([F(t, x, y)]_\alpha, [F(t, x, y)]_\beta) < \varepsilon.$$

In this section, we discuss the solution of problem (2) and the main result is the following existence theorem. For simplicity, we assume that $\Omega \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m$ has the form

$$\Omega = [t_0 - a, t_0 + a] \times B_{R^n}(x_0, b_1) \times B_{R^m}(y_0, b_2)$$

with $a, b_1, b_2 > 0$.

Theorem 4.8. *Suppose that*

(i) fuzzy maps $F : \Omega \to \mathbf{E}^n$ and $G : \Omega \to \mathbf{E}^m$ are both α-level uniformly continuous and fuzzy integrably bounded, $\alpha : \mathbf{R}^n \times \mathbf{R}^m \to [0,1]$ and $\beta : \mathbf{R}^n \times \mathbf{R}^m \to [0,1]$ are uniformly continuous;

(ii) for any $(t,u,x),(t,v,y) \in \Omega$,

$$|t-t_0|H(F(t,u,x),F(t,v,y)) \le k_1\|u-v\| + k_2\|x-y\|$$

and

$$|t-t_0|H(G(t,u,x),G(t,v,y)) \le k_3\|u-v\| + k_4\|x-y\|;$$

(iii) for any $(t,u,x),(t,v,y) \in \Omega$,

$$|t-t_0|^{\tilde{\beta}}H(F(t,u,x),F(t,v,y)) \le A_1\|u-v\|^{\tilde{\alpha}} + A_2\|x-y|^{\tilde{\alpha}}$$

and

$$|t-t_0|^{\tilde{\beta}}H(G(t,u,x),G(t,v,y)) \le A_3\|u-v\|^{\tilde{\alpha}} + A_4\|x-y\|^{\tilde{\alpha}};$$

(iv) $\tilde{\beta}, A_i, k_i > 0$ for $i = 1,2,3,4$, $0 < \tilde{\alpha} < 1$, $\tilde{\beta} < \tilde{\alpha}$, and $k(1-\tilde{\alpha}) < (1-\tilde{\beta})$, where

$$k = \max\{k_1,k_2\} + \max\{k_3,k_4\}.$$

Then there exists at least one solution for problem (2).

Proof. Consider the multivalued map $\widetilde{F}(t,x,y) = [F(t,x,y)]_{\alpha(x,y)}$ and $\widetilde{G}(t,x,y) = [G(t,x,y)]_{\beta(x,y)}$. We claim that $\widetilde{F}$ and $\widetilde{G}$ are both upper semicontinuous. To get this, we need only to discuss the situation of $\widetilde{F}$, as $\widetilde{G}$ is similar. Given a point $(t_0,x_0,y_0) \in \Omega$, the neighborhood of $\widetilde{F}(t_0,x_0,y_0)$ could be written as

$$B(\widetilde{F}(t_0,x_0,y_0),r) = \{v \in \mathbf{R}^n |\ d(v,\widetilde{F}(t_0,x_0,y_0)) \le r\}.$$

For any $(t,x,y) \in \Omega$ and $u \in \widetilde{F}(t,x,y)$, we have

$$\begin{aligned} d(u,\widetilde{F}(t_0,x_0,y_0)) &\le H(\widetilde{F}(t,x,y),\widetilde{F}(t_0,x_0,y_0)) \\ &= H([F(t_0,x_0,y_0)]_{\alpha(x_0,y_0)},[F(t,x,y)]_{\alpha(x,y)}) \\ &\le H([F(t_0,x_0,y_0)]_{\alpha(x_0,y_0)},[F(t,x,y)]_{\alpha(x_0,y_0)}) \\ &\quad + H([F(t,x,y)]_{\alpha(x_0,y_0)},[F(t,x,y)]_{\alpha(x,y)}) \\ &\le H(F(t_0,x_0,y_0),F(t,x,y)) \\ &\quad + H([F(t,x,y)]_{\alpha(x_0,y_0)},[F(t,x,y)]_{\alpha(x,y)}). \end{aligned} \tag{3}$$

From the righthand side of the above inequality (3), we can imply the upper semicontinuity of $\widetilde{F}(t,x,y)$. First, by the condition (*ii*) and condition (*i*), F is α-level uniformly continuous and that $\alpha(x,y)$ is uniformly continuous. Then, one can easily find a small enough neighborhood of (t_0,x_0,y_0), $U \subset \Omega$, such that for any $(t,x,y) \in U$, $u \in \widetilde{F}(t,x,y)$

$$d(u,\widetilde{F}(t_0,x_0,y_0)) \le r,$$

that is, $\widetilde{F}(U) \subset \widetilde{F}(t_0,x_0,y_0)$. Thus, $\widetilde{F}(t,x,y)$ is upper semicontinuous.

It follows from Lemma 4.3 that there exists a real constant $M > 0$ such that

$$\max\{\|\widetilde{F}(t,x,y)\|, \|\widetilde{G}(t,x,y)\|\} \leq M$$

on Ω. We denote the interval $I = [t_0 - h, t_0 + h]$, where $h = \min\{a, \frac{b_1}{M}, \frac{b_2}{M}\}$. We shall prove the existence of a solution of problem (2) on this interval I by an application of Lemma 4.5.

Let (X, d_X) and (Y, d_Y) be two metric spaces defined as follows:

$$X = \{x \in C(I, R^n) \mid \|x(t) - x_0\| \leq b_1, \forall t \in I, x(t_0) = x_0\}$$

with

$$d_X : X \times X \to R^+ \cup \{+\infty\}, \quad d_X(x_1, x_2) := \sup_{t\in I}\left\{\frac{\|x_1(t) - x_2(t)\|}{|t - t_0|^{pk}}\right\},$$

and

$$Y = \{y \in C(I, R^m) \mid \|y(t) - y_0\| \leq b_2, \forall t \in I, y(t_0) = y_0\}$$

with

$$d_Y : Y \times Y \to R^+ \cup \{+\infty\}, \quad d_Y(y_1, y_2) := \sup_{t\in I}\left\{\frac{\|y_1(t) - y_2(t)\|}{|t - t_0|^{pk}}\right\},$$

where

$$p > 1, \quad pk(1 - \tilde{\alpha}) < (1 - \tilde{\beta}).$$

Then we know that (X, d_X) and (Y, d_Y) are two complete generalized metric spaces from [38].

Let

$$\begin{aligned} X \times Y &= \{(x, y) \in C(I, R^n) \times C(I, R^m) \mid \|(x(t) - x_0\| \leq b_1, \|y(t) - y_0\| \leq b_2, \forall t \in I, \\ &\quad x(t_0) = x_0, y(t_0) = y_0\}, \end{aligned}$$

and $d_{X\times Y} : (X \times Y) \times (X \times Y) \to R^+ \cup \{+\infty\}$ be defined by

$$\begin{aligned} & d_{X\times Y}((x_1, y_1), (x_2, y_2)) \\ = \;& d_X(x_1, x_2) + d_Y(y_1, y_2) \\ = \;& \sup_{t\in I}\left\{\frac{\|x_1(t) - x_2(t)\|}{|t - t_0|^{pk}}\right\} + \sup_{t\in I}\left\{\frac{\|y_1(t) - y_2(t)\|}{|t - t_0|^{pk}}\right\}. \end{aligned}$$

Then it is easy to see that $(X \times Y, d_{X\times Y})$ is also a complete generalized metric space.
We define the multivalued operator $T : X \times Y \to X \times Y$ as follows:

$$\begin{aligned} T(x, y) &= \left\{(\bar{x}, \bar{y}) \,\middle|\, \bar{x}(t) \in x_0 + \int_{t_0}^{t} [F(s, x(s), y(s))]_{\alpha(x(s), y(s))} ds \;\; a.e.\ I, \right. \\ &\qquad \left. \bar{y}(t) \in y_0 + \int_{t_0}^{t} [G(s, x(s), y(s))]_{\beta(x(s), y(s))} ds \;\; a.e.\ I \right\}, \end{aligned}$$

where $\int_{t_0}^{t}[F(s, x(s), y(s))]_{\alpha(x(s),y(s))}ds$ and $\int_{t_0}^{t}[G(s, x(s), y(s))]_{\beta(x(s),y(s))}ds$ are multivalued integrals of Aumann [4].

It is obvious that a pair $(x^*(t), y^*(t))$ is a fixed point of T if and only if $(x^*(t), y^*(t))$ is a solution of problem (2).

Now we will prove that T satisfies all the hypotheses of Lemma 4.5.

(i) $T(x, y) \neq \emptyset$ for each $(x, y) \in X \times Y$.

Consider the multivalued operator $\widetilde{F}_{(x,y)}(t) = \widetilde{F}(t, x(t), y(t)) = [F(t, x(t), y(t))]_{\alpha(x(t),y(t))}$, which is also upper semicontinuous with compact values. By the Kuratowski-Ryll-Nardzewski selection theorem [26], $\widetilde{F}_{(x,y)}(t)$ has a measurable selection $f_{(x,y)}(t) \in \widetilde{F}_{(x,y)}(t)$ for all $t \in I$. From condition (i), it is obviously that $f_{(x,y)}(t) \in \widetilde{F}_{(x,y)}(t)$ integrable on I in the sense of Lebesgue-Bochner integral [2, 4, 22]. Let $u(t) = x_0 + \int_{t_0}^{t} f_{(x,y)}(s)ds$. Similarly, we can also find a measurable selection $g_{(x,y)}(t) \in \widetilde{G}_{(x,y)}(t) = [G(t, x(t), y(t))]_{\beta(x(t),y(t))}$ for all $t \in I$. Let $v(t) = x_0 + \int_{t_0}^{t} g_{(x,y)}(s)ds$. Then we obtain $(u(t), v(t)) \in T(x, y)$ and so $T(x, y) \neq \emptyset$.

(ii) $T(x, y)$ is closed for each $(x, y) \in X \times Y$. Suppose that (x_n, y_n) is a sequence in $T(x, y)$ which converges to $(x^*, y^*) \in X \times Y$. Then we know that

$$x_n(t) \in x_0 + \int_{t_0}^{t} \widetilde{F}_{(x,y)}(s)ds \;\; a.e$$

and

$$y_n(t) \in y_0 + \int_{t_0}^{t} \widetilde{G}_{(x,y)}(s)ds \;\; a.e.$$

Since $x_0 + \int_{t_0}^{t} \widetilde{F}_{(x,y)}(s)ds$ and $y_0 + \int_{t_0}^{t} \widetilde{G}_{(x,y)}(s)ds$ are closed [25], it follows that $(x^*, y^*) \in T(x, y)$.

(iii) $T(x, y)$ is a multivalued contraction.

We shall prove that there exists $L \in (0, 1)$ such that, for each $(x_1, y_1), (x_2, y_2) \in X \times Y$ with $d_{X\times Y}((x_1, y_1), (x_2, y_2)) < \infty$,

$$H(T(x_1, y_1), T(x_2, y_2)) \leq L d_{X\times Y}((x_1, y_1), (x_2, y_2)).$$

To get this, let $(x_1, y_1), (x_2, y_2) \in X \times Y$. For each pair $(u_1, v_1) \in T(x_1, y_1)$, which means there exist $f_{(x_1,y_1)}(s) \in \widetilde{F}_{(x_1,y_1)}(s)$ and $g_{(x_1,y_1)}(s) \in \widetilde{G}_{(x_1,y_1)}(s)$ such that

$$\begin{cases} u_1(t) = x_0 + \int_{t_0}^{t} f_{(x_1,y_1)}(s)ds, \\ v_1(t) = y_0 + \int_{t_0}^{t} g_{(x_1,y_1)}(s)ds. \end{cases}$$

From Lemma 4.4, there exists a measurable selection $f_{(x_2,y_2)}(s) \in \widetilde{F}_{(x_2,y_2)}(s)$ which satisfies

$$\begin{aligned} \|f_{(x_2,y_2)}(s) - f_{(x_1,y_1)}(s)\| &\leq H(\widetilde{F}_{(x_1,y_1)}(s), \widetilde{F}_{(x_2,y_2)}(s)) \\ &= H([F(s, x_1(s), y_1(s))]_{\alpha(x_1(s),y_1(s))}, [F(s, x_2(s), y_2(s))]_{\alpha(x_2(s),y_2(s))}) \\ &\leq H(F(s, x_1(s), y_1(s)), F(s, x_2(s), y_2(s))). \end{aligned}$$

Similarly, we can find a measurable selection $g_{(x_2,y_2)}(s) \in \widetilde{G}_{(x_2,y_2)}(s)$ such that

$$\|g_{(x_2,y_2)}(t) - g_{(x_1,y_1)}(s)\| \leq H(G(s, x_1(s), y_1(s)), G(s, x_2(s), y_2(s))).$$

Let

$$\begin{cases} u_2(t) = x_0 + \int_{t_0}^{t} f_{(x_2,y_2)}(s)ds, \\ v_2(t) = y_0 + \int_{t_0}^{t} g_{(x_2,y_2)}(s)ds. \end{cases}$$

Then $(u_2, v_2) \in T(x_2, y_2)$. It follows that

$$\begin{aligned}
\|u_2(t) - u_1(t)\| &= \left\| x_0 + \int_{t_0}^{t} f_{(x_2,y_2)}(s)ds - x_0 - \int_{t_0}^{t} f_{(x_1,y_1)}(s)ds \right\| \\
&\le \int_{t_0}^{t} \|f_{(x_2,y_2)}(s) - f_{(x_1,y_1)}(s)\| ds \\
&\le \int_{t_0}^{t} H(F(s, x_2(s), y_2(s)), F(s, x_1(s), y_1(s)))ds \\
&= \int_{t_0}^{t} \frac{k_1\|x_2(s) - x_1(s)\| + k_2\|y_2(s) - y_1(s)\|}{|s - t_0|} ds \\
&\le \max\{k_1, k_2\} \int_{t_0}^{t} \frac{\|x_2(s) - x_1(s)\| + \|y_2(s) - y_1(s)\|}{|s - t_0|^{pk}} |s - t_0|^{pk-1} ds \\
&\le \max\{k_1, k_2\} d_{X\times Y}((x_1, y_1), (x_2, y_2)) \frac{|t - t_0|^{pk}}{pk}
\end{aligned}$$

Thus,

$$\frac{\|u_2(t) - u_1(t)\|}{|t - t_0|^{pk}} \le \max\{k_1, k_2\} \frac{1}{pk} d_{X\times Y}((x_1, y_1), (x_2, y_2))$$

and so

$$d_X(u_1, u_2) \le \max\{k_1, k_2\} \frac{1}{pk} d_{X\times Y}((x_1, y_1), (x_2, y_2)).$$

Similarly, we have

$$d_Y(v_1, v_2) \le \max\{k_3, k_4\} \frac{1}{pk} d_{X\times Y}((x_1, y_1), (x_2, y_2)).$$

It follows that

$$d_{X\times Y}(u_1, v_1), (u_2, v_2)) \le \frac{1}{p} d_{X\times Y}((x_1, y_1), (x_2, y_2)).$$

Interchanging the roles of (x_1, y_1) and (x_2, y_2), we can get a similar inequality and that

$$H(T(x_1, y_1), T(x_2, y_2)) \le \frac{1}{p} d_{X\times Y}((x_1, y_1), (x_2, y_2)), \quad \forall (x_1, y_1), (x_2, y_2) \in X \times Y.$$

(iv) T admits a sequence of successive approximations , $\varphi_n = (x_n, y_n)_{n\in\mathbf{N}}$ with the property that there exists an index $N \in \mathbf{N}$ such that $d_{X\times Y}(\varphi_N, \varphi_{N+l}) < \infty$ for all $l \in \mathbf{N} \cup \{0\}$.

To get this, let $\varphi_n = (x_n, y_n)_{n\in\mathbf{N}}$ a successive approximations for T(where φ_0 is arbitrary). Let $\varphi_1 \in T(\varphi_0)$. It means that there exist $f_{(x_0,y_0)}(s) \in \widetilde{F}_{(x_0,y_0)}(s)$ and $g_{(x_0,y_0)}(s) \in \widetilde{G}_{(x_0,y_0)}(s)$ such that

$$\begin{cases} x_1(t) = x_0 + \int_{t_0}^{t} f_{(x_0,y_0)}(s)ds, \\ y_1(t) = y_0 + \int_{t_0}^{t} g_{(x_0,y_0)}(s)ds. \end{cases}$$

Letting $\varphi_2 \in T(\varphi_1)$, by the definition of T, we obtain again that there exist a $f_{(x_1,y_1)}(s) \in \widetilde{F}_{(x_1,y_1)}(s)$ and a $g_{(x_1,y_1)}(s) \in \widetilde{G}_{(x_1,y_1)}(s)$ such that

$$\begin{cases} x_2(t) = x_0 + \int_{t_0}^{t} f_{(x_1,y_1)}(s)ds, \\ y_2(t) = y_0 + \int_{t_0}^{t} g_{(x_1,y_1)}(s)ds. \end{cases}$$

By the boundedness of $\widetilde{F}$ and $\widetilde{G}$ we have

$$\begin{aligned}\|x_2(t)-x_1(t)\| &\le \left\|\int_{t_0}^{t} f_{(x_1,y_1)}(s)ds-\int_{t_0}^{t} f_{(x_0,y_0)}(s)ds\right\|\\ &\le \int_{t_0}^{t}\|f_{(x_1,y_1)}(s)-f_{(x_0,y_0)}(s)\|ds\\ &\le 2M|t-t_0|\end{aligned}$$

and

$$\begin{aligned}\|y_2(t)-y_1(t)\| &\le \left\|\int_{t_0}^{t} g_{(x_1,y_1)}(s)ds-\int_{t_0}^{t} g_{(x_0,y_0)}(s)ds\right\|\\ &\le \int_{t_0}^{t}\|g_{(x_1,y_1)}(s)-g_{(x_0,y_0)}(s)\|ds\\ &\le 2M|t-t_0|.\end{aligned}$$

According to Lemma 2.4 we can find a measurable pair $(f_{(x_2,y_2)}(s), g_{(x_2,y_2)}(s))$ with $f_{(x_2,y_2)}(s)\in\widetilde{F}_{(x_2,y_2)}(s)$ and $g_{(x_2,y_2)}(s)\in\widetilde{G}_{(x_2,y_2)}(s)$ such that

$$\|f_{(x_2,y_2)}(s)-f_{(x_1,y_1)}(s)\|\le H(F(s,x_1(s),y_1(s)),F(s,x_2(s),y_2(s)))$$

and

$$\|g_{(x_2,y_2)}(t)-g_{(x_1,y_1)}(s)\|\le H(G(s,x_1(s),y_1(s)),G(s,x_2(s),y_2(s))).$$

Let

$$\begin{cases} x_3(t)=x_0+\int_{t_0}^{t} f_{(x_2,y_2)}(s)ds,\\ y_3(t)=y_0+\int_{t_0}^{t} g_{(x_2,y_2)}(s)ds.\end{cases}$$

Then it is easy to see that $(x_3,y_3)\in T(x_2,y_2)$. By the assumptions, we have

$$\begin{aligned}\|x_3(t)-x_2(t)\| &\le \int_{t_0}^{t}\|f_{(x_2,y_2)}(s)-f_{(x_1,y_1)}(s)\|ds\\ &\le \int_{t_0}^{t} H(F(s,x_2(s),y_2(s)),F(s,x_1(s),y_1(s)))ds\\ &\le \int_{t_0}^{t}\frac{A_1\|x_2(s)-x_1(s)\|^{\tilde\alpha}+A_2\|y_2(s)-y_1(s)\|^{\tilde\alpha}}{|s-t_0|^{\tilde\beta}}ds\\ &\le (A_1+A_2)(2M)^{\tilde\alpha}\int_{t_0}^{t}|s-t_0|^{\tilde\alpha-\tilde\beta}ds\\ &\le (A_1+A_2)(2M)^{\tilde\alpha}\frac{|t-t_0|^{1+\tilde\alpha-\tilde\beta}}{1+\tilde\alpha-\tilde\beta}\\ &\le (A_1+A_2)(2M)^{\tilde\alpha}|t-t_0|^{1+\tilde\alpha-\tilde\beta}.\end{aligned}$$

Similarly,

$$\|y_3(t)-y_2(t)\|\le (A_3+A_4)(2M)^{\tilde\alpha}|t-t_0|^{1+\tilde\alpha-\tilde\beta}.$$

Thus, we have

$$\begin{aligned}\|\varphi_3(t)-\varphi_2(t)\| &= \|x_3(t)-x_2(t)\|+\|y_3(t)-y_2(t)\|\\ &\le (A_1+A_2)(2M)^{\tilde\alpha}|t-t_0|^{1+\tilde\alpha-\tilde\beta}+(A_3+A_4)(2M)^{\tilde\alpha}|t-t_0|^{1+\tilde\alpha-\tilde\beta}\\ &\le A(2M)^{\tilde\alpha}|t-t_0|^{1+\tilde\alpha-\tilde\beta},\end{aligned}$$

where

$$A = \max\{A_1 + A_2, A_3 + A_4\}.$$

Generally, we can get

$$\begin{aligned}
\|\varphi_{n+1}(t) - \varphi_n(t)\| &= \|x_{n+1}(t) - x_n(t)\| + \|y_{n+1}(t) - y_n(t)\| \\
&\leq A^{1+\tilde{\alpha}^1+\cdots+\tilde{\alpha}^{n-2}}(2M)^{\tilde{\alpha}^{n-1}}|t - t_0|^{(1-\tilde{\beta})(1+\tilde{\alpha}+\cdots+\tilde{\alpha}^{n-2})+\tilde{\alpha}^{n+1}} \\
&< B|t - t_0|^{(1-\tilde{\beta})(1+\tilde{\alpha}^1+\cdots+\tilde{\alpha}^{n-2})+\tilde{\alpha}^{n-1}},
\end{aligned}$$

where

$$B = A^{\frac{1}{1-\tilde{\alpha}}} \max\{2M, 1\}.$$

In view of $pk(1 - \tilde{\alpha}) < 1 - \tilde{\beta}$ there exists an index $N \in \mathbf{N}$ such that

$$(1 - \tilde{\beta})(1 + \tilde{\alpha}^1 + \cdots + \tilde{\alpha}^{n-2}) + \tilde{\alpha}^{n-1} > pk,$$

for each $n \geq N$, and so

$$\frac{\|\varphi_{n+1}(t) - \varphi_n(t)\|}{|t - t_0|^{pk}} \leq B|t - t_0|^{\eta_n},$$

for all $n \geq N$, where

$$\eta_n = (1 - \tilde{\beta})(1 + \tilde{\alpha}^1 + \cdots + \tilde{\alpha}^{n-2}) + \tilde{\alpha}^{n-1} - pk > 0.$$

Thus, for all $n \geq N$,

$$d_{X\times Y}(\varphi_{n+1}, \varphi_n) < \infty.$$

This completes the proof. □

Remark 4.9. *Theorem 4.8 extends Theorem 3.1 of Petrusel* [40] *from classical multivalued Cauchy problem to the system of fuzzy differential inclusions.*

Remark 4.10. *Let $\alpha(x, y)$ and $\beta(x, y)$ be a constant α, and let*

$$F(t, x(t), y(t)) = G(t, x(t), y(t)) = \widetilde{F}(t, x(t)).$$

By Theorem 4.8, we know that problem (2) has a solution set $X_\alpha(t)$ with $\alpha \in [0, 1]$. Under the same conditions of Theorem 6.2.3 in [31]*, it follows from Lemma 2.2 that the family $\{X_\alpha(t)\}_{\alpha\in[0,1]}$ is the family of α-level sets of the solution for the following fuzzy initial value problem*

$$\begin{cases} x'(t) \in \widetilde{F}(t, x(t)), \\ x(t_0) = X_0. \end{cases}$$

Thus, Theorem 4.8 is actually an extension of Proposition 4 of Hüllermeier [23] from a fuzzy initial value problem to the system of fuzzy differential inclusions.

Next we give the following example to show an application of Theorem 4.8.

Example 4.11. *In* [35]*, Liu et al. considered the following polytopic differential inclusion systems:*

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = [\alpha(1 + x_2^2(t)) + (1 - \alpha)](u_1(t) + \sin t), \\ \dot{x}_4(t) = \alpha x_1^2(t) + (1 - \alpha)x_2(t) + [\alpha(1 + x_2^2(t)) + (1 - \alpha)]u_2(t). \end{cases}$$

This is a nonlinear control system with an uncertain parameter $\alpha \in [0,1]$, which can be rewritten as

$$\begin{cases} \dot{\bar{x}}(t) = A_{11}\bar{x}(t) + A_{12}\bar{y}(t), \\ \dot{\bar{y}}(t) \in co\{f_i(\bar{x},\bar{y}) + g_i(\bar{x},\bar{y})B(u(t)+w(t)), i = 1,2\}, \end{cases}$$

where

$$A_{11} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\bar{x}(t) = (x_1(t), x_2(t))^T$, $\bar{y}(t) = (x_3(t), x_4(t))^T$ *are the system states,*

$$f_1(\bar{x},\bar{y}) = \begin{pmatrix} 0 \\ x_1^2 \end{pmatrix}, \quad f_2(\bar{x},\bar{y}) = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \quad g_1(\bar{x},\bar{y}) = 1 + x_2^2, \quad g_2(\bar{x},\bar{y}) = 1,$$

$u(t) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ *is the controlled variable,* $w(t) = \begin{pmatrix} \sin t \\ 0 \end{pmatrix}$ *is the bounded disturbance and* $co\{\cdot\}$ *is convex hull of a set.*

The authors of [35] *discussed the solutions and the performance of this control problem in detail for different* α*. Analogously, if the system parameters are linguistic or effected by the modeler's subjective experience, it can be transformed to a fuzzy differential inclusion system aforementioned in this paper. For simplicity, let the uncertain parameter* α *in the above equations be 1,* $\alpha(x,y)$ *and* $\beta(x,y)$ *in (2) be some constant* α *and* $\beta \in [0,1]$*, the controlled variable* $u(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$*. Thus, in the fuzzy differential inclusion system (2), we have*

$$F(t,\bar{x},\bar{y}) = A_{11}\bar{x} + A_{12}\bar{y}, \quad G(t,\bar{x},\bar{y}) = f_1(\bar{x},\bar{y}) + g_1(\bar{x},\bar{y})B(u(t)+w(t))$$

and so (2) turns to be

$$\begin{cases} \dot{\bar{x}}(t) \in [F(t,\bar{x}(t),\bar{y}(t))]_\alpha = [A_{11}]\alpha\bar{x} + [A_{12}]\alpha\bar{y}, \\ \dot{\bar{y}}(t) \in [G(t,\bar{x}(t),\bar{y}(t))]_\beta = f_1(\bar{x},\bar{y}) + g_1(\bar{x},\bar{y})[B]\beta(u(t)+w(t)), \\ \bar{x}(t_0) = \bar{x}_0, \bar{y}(t_0) = \bar{y}_0, \end{cases}$$

where A_{11}*,* A_{12} *and B are the matrices of fuzzy numbers, the arithmetic operations of which are generalized by Extension Principle [47]. By the boundedness of* Ω *and the construction of the system, it is easy to verify that for any* $\alpha,\beta \in [0,1]$*,* $[F]\alpha$ *and* $[G]\beta$ *satisfy the Lipschitz condition, which means that for any* $(t,u,x),(t,v,y) \in \Omega$*, there exists some* $L_1, L_2, L_3, L_4 > 0$ *such that*

$$H(F(t,u,x),F(t,v,y)) \le L_1\|u-v\| + L_2\|x-y\|$$

and

$$H(G(t,u,x),G(t,v,y)) \le L_3\|u-v\| + L_4\|x-y\|.$$

Thus the conditions of Theorem 4.8 are naturally satisfied.

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